

IX. OPTICAL PULSE PROPAGATION

THE ELECTROMAGNETIC NONLINEAR SCHRÖDINGER EQUATION:

We begin our discussion of optical pulse propagation³⁵ with a derivation of the nonlinear Schrödinger (NLS) equation. To that end, we recall Equations [VII-23] and [VII-23] from the early lecture set entitled *Nonlinear Optics I* -- i.e.

$$\nabla^2 \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) - \frac{1}{c^2} \frac{\partial^2 \bar{\mathbf{E}}(\bar{\mathbf{r}}, t)}{\partial t^2} = -\mu_0 \nabla \cdot \bar{\mathbf{P}}^{(\text{NL})}(\bar{\mathbf{r}}, t) \quad [\text{IX-1}]$$

$$\nabla \cdot \left[\epsilon_0 \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) \right] = -\nabla \cdot \bar{\mathbf{P}}^{(\text{NL})}(\bar{\mathbf{r}}, t) \quad [\text{IX-2}]$$

In this treatment we will confine our attention to wave propagation in **uniform, isotropic optical materials** -- viz., glass fibers. For such materials, we can write

$$\bar{\mathbf{P}}^{(\text{NL})}(\bar{\mathbf{r}}, t) = \epsilon_0 \chi_{\text{NL}}(\bar{\mathbf{r}}, t) \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) \quad [\text{IX-3}]$$

where $\chi_{\text{NL}}(\bar{\mathbf{r}}, t) = \frac{3}{4} \epsilon_0 \chi_{\text{xxx}}^{(3)} |\bar{\mathbf{E}}(\bar{\mathbf{r}}, t)|^2$ and, thus, Equation [IX-1] simplifies to

$$\nabla^2 \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) + \left(k_0^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left[\epsilon_0 \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) + \chi_{\text{NL}}(\bar{\mathbf{r}}, t) \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) \right] = 0 \quad [\text{IX-4}]$$

where $k_0 = \omega/c$.³⁶

To proceed, postulate that this nonlinear Helmholtz equation can be treated by *separation of variables* methods. In particular, we are looking for a time-localized solution (a

³⁵ An excellent reference on this subject is Govind P. Agrawal's *Nonlinear Fiber Optics*, Academic Press (1989) ISBN 0-12-045140-9.

³⁶ In this simplification, we have taken $\nabla \cdot \bar{\mathbf{E}}(\bar{\mathbf{r}}, t) = -\left[\epsilon_0 - \chi_{\text{NL}}(\bar{\mathbf{r}}, t) \right]^{-1} \nabla \cdot \bar{\mathbf{P}}^{(\text{NL})}(\bar{\mathbf{r}}, t) = 0$.

pulse) with a relatively narrow frequency spectrum (or “group” of frequencies centered on a frequency ω_{ctr} . Thus, we assume a separation of variables solution

$$\vec{E}(\vec{r}, t) = F(x, y) \vec{G}(z, t - t_{ctr}) \exp(-i \omega_{ctr} z) \quad [IX-5]$$

where ω_{ctr} is a wave or propagation number to be associated with ω_{ctr} and, thus, Equation [IX-4] becomes

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [k^2 - \omega_{ctr}^2] F = 0 \quad \vec{G} + \frac{\partial^2 \vec{G}}{\partial z^2} - i 2 \omega_{ctr} \frac{\partial \vec{G}}{\partial z} + [\omega_{ctr}^2 - k^2] \vec{G} = 0 \quad [IX-6]$$

where $k = k_0 \sqrt{[\epsilon(\vec{r}, t) + \epsilon_{NL}(\vec{r}, t)]/\epsilon_0}$. In the linear problem ω_{ctr}^2 would be the “separation constant,” but in this case we will need a bit more elaboration. Nevertheless, we shall assume that we can find a set of functions $F(x, y)$ and values ω_{ctr}^2 that satisfy the equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [k^2 - \omega_{ctr}^2] F = 0 \quad [IX-7a]$$

so that

$$\frac{\partial^2 \vec{G}}{\partial z^2} - i 2 \omega_{ctr} \frac{\partial \vec{G}}{\partial z} + [\omega_{ctr}^2 - k^2] \vec{G} = 0 \quad [IX-7b]$$

To use perturbation theory, we first reduce Equation [IX-7a] to a solvable linear problem by writing

$$k^2 = k_0^2 [\epsilon(\vec{r}, t) + \epsilon_{NL}(\vec{r}, t)]/\epsilon_0 = [n^2(\vec{r}, t) + \epsilon_{NL}(\vec{r}, t)] k_0^2 \quad [IX-8a]$$

$$\omega_{ctr}^2 = [\omega_{ctr}^2 + \epsilon_{NL}(\vec{r}, t)] \quad [IX-8b]$$

where $n(\vec{r}) = \frac{n_{NL}(\vec{r})}{2n(\vec{r})} = \frac{3}{8} \frac{n_{xxx}^{(3)}(\vec{r})}{n(\vec{r})} |\vec{E}(\vec{r})|^2$. Thus, to first order we need to solve the linear equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [n^2 k_0^2 - \beta^2] F = 0. \quad [IX-9]$$

which taken together with appropriate boundary conditions defines the linear eigenvalue problem for propagation in the medium where the functions F are the eigenfunctions and the values β are the eigenvalues. In the previous lecture set -- *i.e.*, VIII. Guided Waves in Planar Structures -- we found a set of eigenfunctions and eigenvalues appropriate to the dielectric-slab guidewave propagation

Following an earlier discussion, we now presume that $\left| \frac{\partial^2 \vec{G}}{\partial z^2} \right| \ll \left| 2 \frac{\partial \vec{G}}{\partial z} \right|$ -- *i.e.*, we take the ***slowly vary amplitude*** or ***paraxial*** approximation -- so that Equation [IX-7b] reduces to

$$-i 2 \frac{\partial \vec{G}}{\partial z} + \left[-\beta^2 - \frac{\partial^2}{\partial z^2} \right] \vec{G} = 0 \quad [IX-10a]$$

where $-\beta^2 - \frac{\partial^2}{\partial z^2} = -\beta_{ctr}^2 - \frac{\partial^2}{\partial z^2} + 2$ and, thus,

$$-i 2 \frac{\partial \vec{G}}{\partial z} + \left[-\beta_{ctr}^2 - \frac{\partial^2}{\partial z^2} \right] \vec{G} + 2 \vec{G} = 0 \quad [IX-10b]$$

For equation [IX-7a] to be completely satisfied in first order, we must have

$$\begin{aligned}
&= \frac{n k_0^2}{8} \frac{\int \int |F(x,y)|^2 dx dy}{\int \int |F(x,y)|^2 dx dy} \\
&= \frac{3 k_0^2}{8} \int \int \int \int \left| \vec{G}(z, - \text{ctr}) \right|^2 \frac{|F(x,y)|^4 dx dy}{|F(x,y)|^2 dx dy} \quad [IX-11]
\end{aligned}$$

Since we are assuming that propagating pulse has a relatively narrow frequency spectrum, it is reasonable to use a Taylor expansion around ctr for $\left(\right)_{\text{ctr}}$ -- viz.

$$\left(\right)_{\text{ctr}} + \left(- \text{ctr} \right)_1 + \frac{1}{2!} \left(- \text{ctr} \right)_2^2 + \frac{1}{3!} \left(- \text{ctr} \right)_3^3 + \dots \quad [IX-12]$$

where $\left. \frac{d^\ell \left(\right)}{d^\ell} \right|_{=c} \cdot$

Near the **dispersion minimum** in glass fibers (*i.e.* $1.3-1.6\mu m$) we may, to very good approximation, stop with the quadratic term and write

$$\left(\right)_2 - \left(\right)_{\text{ctr}}^2 \left[- \text{ctr} \right] = 2 \left(- \text{ctr} \right)_1 + \frac{1}{2!} \left(- \text{ctr} \right)_2^2 \quad [IX-13]$$

In this approximation, Equation [IX-10] becomes

$$\frac{\vec{G}}{z} + i \left(- \text{ctr} \right)_1 + \frac{1}{2!} \left(- \text{ctr} \right)_2^2 \vec{G} + \frac{i}{\text{ctr}} \vec{G} = 0 \quad [IX-14a]$$

If we take

$$= \frac{3 k_0^2}{8} \int \int \int \int \frac{|F(x,y)|^4 dx dy}{|F(x,y)|^2 dx dy} \quad [IX-15]$$

Equation [IX-10] in the **frequency domain** reduces to

$$\frac{\vec{G}}{z} + i \left(- \quad \right)_1 + \frac{1}{2!} \left(- \quad \right)_2 \vec{G} = -i \left| \vec{G} \right|^2 \vec{G} . \quad [IX-14b]$$

which implies the following **time domain equation** for the pulse envelope:

$$\frac{\vec{G}(z,t)}{z} + \frac{1}{t} \frac{\vec{G}(z,t)}{t} - \frac{i}{2} \frac{2\vec{G}(z,t)}{t^2} + \frac{1}{2} \vec{G}(z,t) = -i \left| \vec{G}(z,t) \right|^2 \vec{G}(z,t) \quad [IX-16]$$

The last term on the left hand side has been added to incorporate the effects of various possible loss mechanisms.

Next we transform into a coordinate system which moves with the “group” -- what might be called the *surfer's coordinates* of the pulse -- i.e., $\{t, z\} \rightarrow \{ \quad , \quad \}$ where $\quad = t - t_0 - \quad_1(z - z_0)$ and $\quad = z - z_0 = z - \quad_1^{-1} t_0$. In term of these surfer's coordinates, Equation [IX-16] becomes³⁷

$$\frac{\vec{G}(\quad, \quad)}{z} = -\frac{1}{2} \vec{G}(\quad, \quad) + \frac{i}{2} \frac{2\vec{G}(\quad, \quad)}{t^2} - i \left| \vec{G}(\quad, \quad) \right|^2 \vec{G}(\quad, \quad) \quad [IX-17]$$

Obviously, if we omit all of the terms on the right hand side of this equation so that $\frac{\vec{G}(\quad, \quad)}{z} = 0$, we would have the ideal situation wherein a **pulse of any shape propagates forever without changing shape at a velocity** $v_g = \quad_1^{-1}$ --the group

³⁷ Since

$$\begin{aligned} \frac{\partial \vec{G}(z,t)}{\partial z} &= -\vec{G}(\quad, \quad) \frac{\partial}{\partial z} + \frac{\partial \vec{G}(\quad, \quad)}{\partial z} = -\vec{G}(\quad, \quad) (1) + \frac{\partial \vec{G}(\quad, \quad)}{\partial z} (-v_g^{-1}) \\ \frac{\partial \vec{G}(z,t)}{\partial t} &= -\vec{G}(\quad, \quad) \frac{\partial}{\partial t} + \frac{\partial \vec{G}(\quad, \quad)}{\partial t} = -\vec{G}(\quad, \quad) (0) + \frac{\partial \vec{G}(\quad, \quad)}{\partial t} (1) \end{aligned}$$

velocity. In treating the less-than-ideal situation, we will initially neglect the effects of the loss (first) term and confine our attention to the competing effects of the **dispersion** (second) and **nonlinear** (third) term. We cast the lossless version of Equation [IX-16] into a normalized, standard form by introducing

$$u(Z, T) = \frac{\tilde{G}(\cdot, \cdot)}{\sqrt{P_0}}, \quad Z = \frac{L}{L_D} = \frac{|z|}{L_D}, \quad T = \sqrt{2} \frac{t}{L_D} \quad \text{---}$$

where L_D is the width of the pulse and P_0 is its peak power.

Thus, we, at last, obtain the standard form of the **nonlinear Schrödinger (NLS) equation**

$$i \frac{u(Z, T)}{Z} - \text{sgn}(z) \frac{u^2(Z, T)}{T^2} + N^2 |u(Z, T)|^2 u(Z, T) = 0 \quad \text{[IX-18]}$$

where $N^2 = L_D/L_{NL} = P_0 / |z|$.

PULSE SOLUTIONS OF LINEAR SCHRÖDINGER EQUATION

If in Equation [IX-10] we set $\tilde{G}(z, t) = U(z, t)$ and neglect the loss and nonlinear terms, we see that $U(z, t)$ satisfies the following differential equation:

$$-\frac{\partial^2 U(z, t)}{\partial z^2} + v_g^{-1} \frac{\partial U(z, t)}{\partial t} = i b \frac{\partial^2 U(z, t)}{\partial t^2} \quad \text{[IX-19]}$$

For **minimal dispersion** -- viz. if $b = \frac{1}{2!} \left. \frac{d^2}{d\omega^2} \right|_{\omega_0} = 0$ -- Equation [IX-19]

becomes

$$-\frac{\partial^2 U(z, t)}{\partial z^2} + v_g^{-1} \frac{\partial U(z, t)}{\partial t} = 0 \quad \text{[IX-20]}$$

which is the basic wave equation for **minimally dispersive media** with the general solution

$$U(z, t) = U(z - v_g t) \quad [IX-21]$$

where $v_g = \left. \frac{d(\omega)}{dk} \right|_{k_0}^{-1}$ is the **group velocity** of the pulse.

When limitation to **first order dispersion** is an adequate approximation, Equation [IX-19] expressed in surfer's coordinates becomes

$$b \frac{\partial^2}{\partial z^2} U(z, t) + i \frac{\partial}{\partial t} U(z, t) = 0 \quad [IX-22]$$

where $b = c^2/2$.

Amazingly, this pulse dispersion equation is the, so called, parabolic equation that we saw earlier in connection with beam propagation -- viz. the paraxial wave propagation equation.

Solution of Pulse Dispersion Equation

For convenience, we restate here Equation [IX-22] the first-order pulse dispersion equation -- viz.

$$b \frac{\partial^2}{\partial z^2} U(z, t) + i \frac{\partial}{\partial t} U(z, t) = 0 .$$

Let us write a Fourier transform for this modulation in terms of "surfer time" -- i.e.

$$U(\omega, z) = \int_{-\infty}^{+\infty} U(\omega, 0) \exp(i k z) d\omega \quad [IX-23a]$$

where

$$U(\omega, 0) = \frac{1}{2} \int_{-\infty}^{+\infty} U(\omega, z) \exp(-i k z) dz. \quad [IX-23b]$$

Thus, the pulse dispersion equation reduces to an ordinary differential equation -- viz.

$$i \frac{d}{dz} U(\omega, z) = b^2 U(\omega, z) \quad [IX-24]$$

for the Fourier transform and thus we have the simple solution

$$U(\omega, z) = U(\omega, 0) \exp(-i b^2 z). \quad [IX-25]$$

Thus, we see that the dispersion changes the phase of **each spectral component** of the pulse by an amount that depends on the frequency and the propagated distance. The general solution may be written as

$$U(\omega, z) = \int_{-\infty}^{+\infty} U(\omega, 0) \exp[i(\omega - \omega_0) z] d\omega \quad [IX-26a]$$

where

$$U(\omega, 0) = \frac{1}{2} \int_{-\infty}^{+\infty} U(\omega, t - t_0) \exp[-i(\omega - \omega_0)(t - t_0)] d(t - t_0). \quad [IX-26b]$$

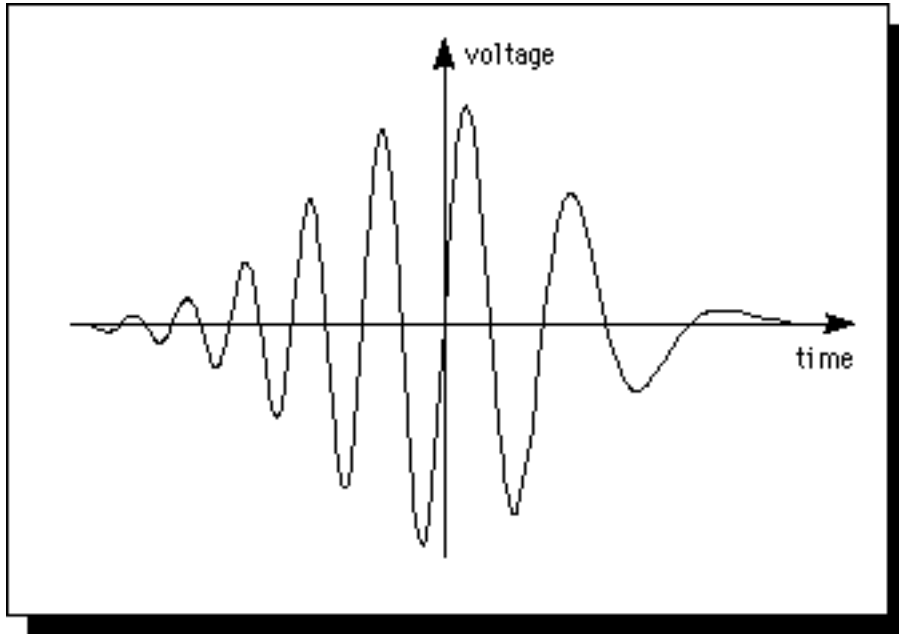
Let us suppose that we have a "chirped" Gaussian at $\omega = 0$ -- i.e.

$$U(0, t - t_0) = \exp - \frac{1 - iC}{2} \frac{(t - t_0)^2}{0} \quad [IX-27a]$$

so that

$$U(0,) = \sqrt{\frac{2}{2(1 - iC)}} \exp - \frac{2}{2(1 - iC)} \quad [IX-27b]$$

where $C > 0$ characterizes an "up-chirp" and $C < 0$ a "down-chirp" pulse.



A Gaussian pulse with a frequency "down-chirp"

Inverting the transform, we see that

$$U(z, t) = \frac{U_0}{\sqrt{1 + i \frac{2b}{L_D} (1 - iC)}} \exp \left[-\frac{(1 - iC)^2}{2 \left[1 + i \frac{2b}{L_D} (1 - iC) \right]} \right] \quad [\text{IX-28a}]$$

or rationalizing

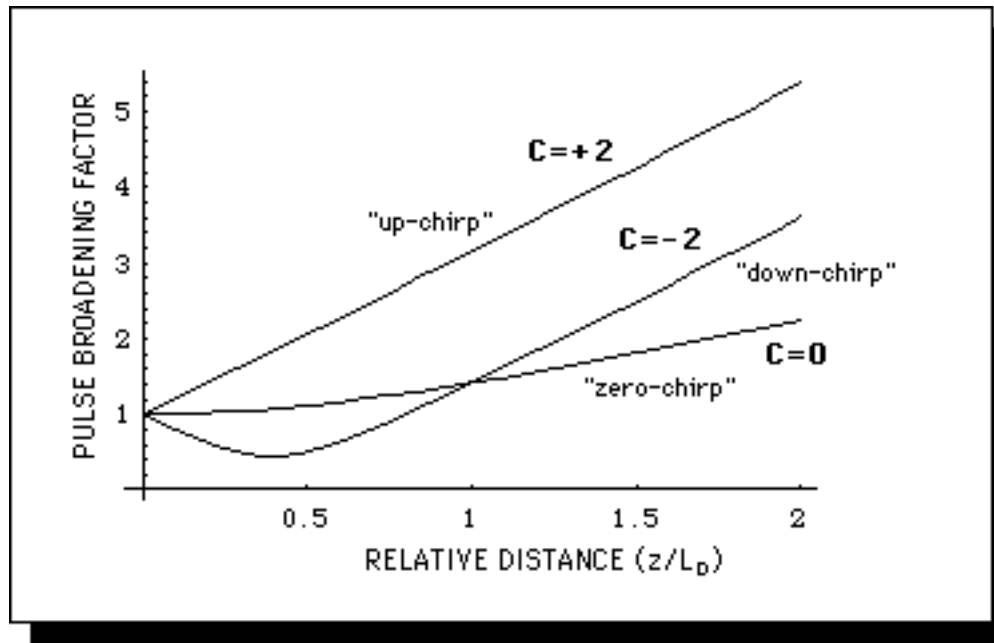
$$U(z, t) = \frac{U_0}{\sqrt{1 + i \frac{2b}{L_D} (1 - iC)}} \exp \left[-\frac{(1 - iC)^2}{2 \left[1 + \frac{2bC}{L_D} + 1 + \frac{2b}{L_D} \right]} \right] \quad [\text{IX-28b}]$$

$$\times \exp \left[\frac{i \frac{2bC}{L_D} (1 + C^2)}{2 \left[1 + \frac{2bC}{L_D} + 1 + \frac{2b}{L_D} \right]} \right]$$

Hence, the **pulse width broadening factor** at a given position z is given by

$$\frac{U(z, t)}{U_0} = \sqrt{1 + \frac{2bC}{L_D} + \frac{2b}{L_D}} = \sqrt{1 + \frac{C^2}{L_D} + \frac{1}{L_D}} \quad [\text{IX-29}]$$

where $L_D = \frac{2}{L_D}$.



The spatial evolution of the pulse width of chirped Gaussian pulse

(For "normal dispersion" -- *i.e.* $\frac{\omega^2}{2} > 0$)

Pulse Solutions of "Negligible Dispersion" Nonlinear Schrödinger Equation

If group velocity dispersion can be neglected, Equation [IX-17] reduces to

$$\frac{\partial \tilde{G}(\omega, z)}{\partial z} = -\frac{\omega^2}{2} \tilde{G}(\omega, z) - i |\tilde{G}(\omega, z)|^2 \tilde{G}(\omega, z) \quad [IX-30]$$

and if we take $\tilde{G}(\omega, z) = U(\omega, z) \sqrt{P_0} \exp(-i\omega^2 z/2)$ we obtain

$$\frac{\partial U(\omega, z)}{\partial z} = -i L_{NL}^{-1} \exp(-i\omega^2 z/2) |U(\omega, z)|^2 U(\omega, z) \quad [IX-31]$$

where the characteristic nonlinear length is given by $L_{NL} = (P_0)^{-1}$. A solution to this equation is readily obtain in the form

$$U(z, t) = U(0, t) \exp[i\phi_{NL}(z, t)] \quad [IX-32a]$$

where

$$\begin{aligned} \phi_{NL}(z, t) &= -[1 - \exp(-z/L_{NL})] [L_{NL}]^{-1} \exp(-z/L_{NL}) |U(0, t)|^2 \\ &= -[z_{eff}/L_{NL}] |U(0, t)|^2 \end{aligned} \quad [IV-32b]$$

and

$$z_{eff} = [1 - \exp(-z/L_{NL})]^{-1} \quad [IX-32c]$$

This interesting result show that so called “self-phase modulation” or SPM gives rise to an **intensity dependent phase shifted or chirped pulse which remains constant in shape** as it propagates. The instantaneous optical frequency shift is given by

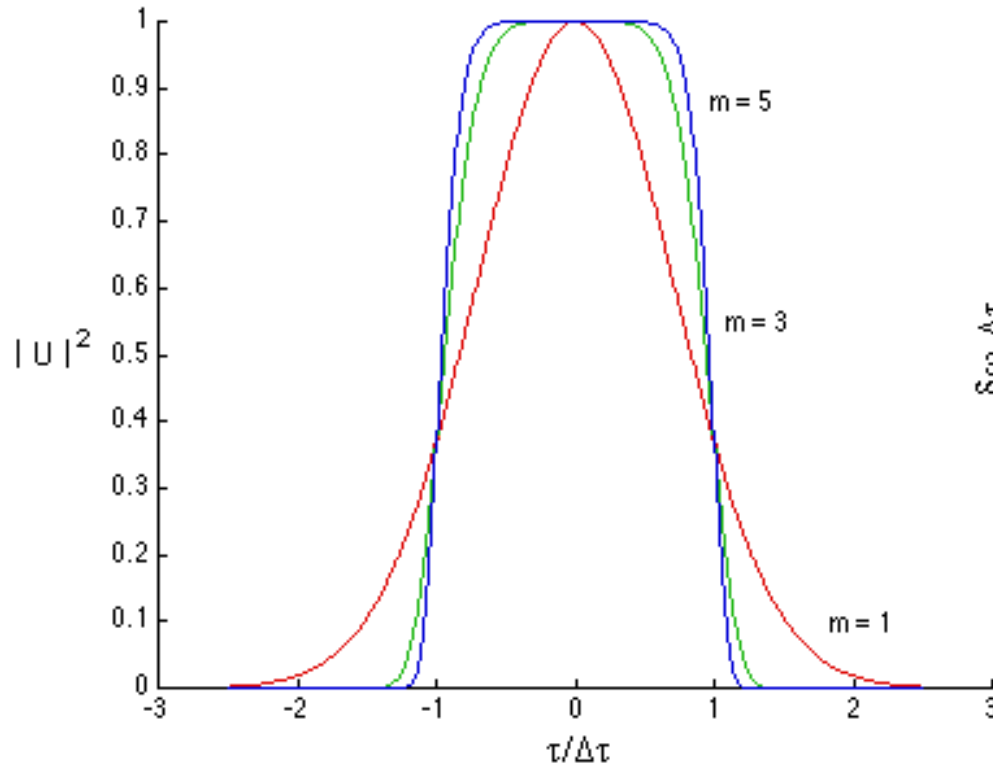
$$\omega(z, t) = \omega(0, t) + \frac{d\phi_{NL}}{dt} = \omega(0, t) - [z_{eff}/L_{NL}] \frac{d|U(0, t)|^2}{dt} \quad [IV-33]$$

Note that the pulse spectrum is “red-shifted” on the leading edge of a pulse and “blue-shifted” on the trailing edge of the pulse. If we suppose the initial pulse to be a super-Gaussian of mth-order -- *i.e.*,

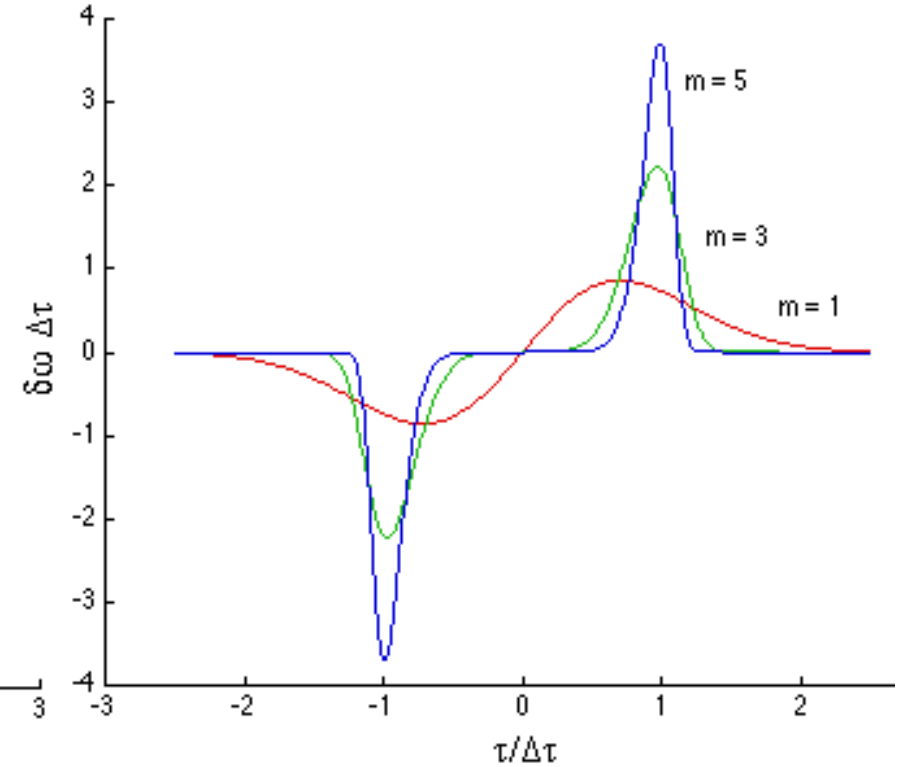
$$U_m(0, t) = \exp\left[-\frac{1+iC}{2} t^{2m}\right] \quad [IV-34]$$

-- then the instantaneous optical frequency shift would be given by

$$\omega(z, t) = \omega(0, t) - \frac{z_{eff}}{L_{NL}} \frac{2m}{t^{2m-1}} \exp\left[-\frac{1+iC}{2} t^{2m}\right] \quad [IV-35]$$



Super-Gaussian Envelope Shapes



SPM Induced Frequency Chirp

A Brief Discussion of Solitons

We return briefly to Equation [IX-18] -- the standard form of the **nonlinear Schrödinger (NLS) equation** -- *i.e.*,

$$i \frac{u(Z,T)}{Z} - \text{sgn}\left(\frac{1}{2}\right) \frac{\partial^2 u(Z,T)}{\partial T^2} + N^2 |u(Z,T)|^2 u(Z,T) = 0 \quad [\text{IX-18}]$$

where $u(Z, T) = \frac{\vec{G}(\cdot, \cdot)}{\sqrt{P_0}}$, $Z = \frac{L}{L_D} = \frac{L}{\frac{1}{2} \lambda_D}$, $T = \sqrt{2} \frac{L}{L_D}$ and $N^2 = L_D/L_{NL} = \frac{P_0}{P_{cr}} \left| \frac{1}{2} \lambda_D \right|$.

In soliton analysis the most common form of NLS equation is the following:

$$i \frac{U(Z, T)}{Z} + \frac{\partial^2 U(Z, T)}{\partial T^2} + |U(Z, T)|^2 U(Z, T) = 0 \quad [\text{IX-36}]$$

where $U(Z, T) = N u(Z, T) \vec{G}(\cdot, \cdot) \sqrt{\frac{1}{2} \lambda_D}$.

If $N = 1$ the following ***fundamental soliton*** will propagate undistorted for an arbitrary distance:

$$U(Z, T) = \text{sech}(T) \exp(i Z/2) \quad [\text{IX-37}]$$

If $N = 2$ the following ***second-order soliton*** will propagate undistorted for an arbitrary distance:

$$U(Z, T) = \frac{4 [\cosh(3T) + 3 \exp(i 4 Z) \cosh(T)]}{[\cosh(4T) + 4 \cosh(2T) + 3 \cos(4 Z)]} \exp(i Z/2) \quad [\text{IX-37}]$$

$$|U(Z, T)|^2 = \frac{16 [\cosh^2(3T) + 9 \cosh^2(T) + 6 \cos(4 Z) \cosh(3T) \cosh(T)]}{[\cosh(4T) + 4 \cosh(2T) + 3 \cos(4 Z)]^2} \quad [\text{IX-37}]$$

TWO SOLITON COLLISION ILLUSTRATIONS

The following are time-lapsed illustration of the propagation and collision of two solitons -- plot here is

$$u(x, t) = 12 \frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}$$

